Math 775: Algebraic Geometry

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About This Course

This course was taken in the Spring of 2022 at UNC Chapel Hill taught by Professor Shrawan Kumar. We used Shaferevich and Hartshorne. These notes were copied from the ones in my notebook and any mistakes are mine and not the lecturers.

Contents

1 Affine Varieties

Let k be any algebraically closed field, we take $\mathbb{A}^n = k^n$ to be Cartesian n-space.

Definition (Zariski Topology)**.** Let $R_n = k[x_1, ..., x_n]$, and $\mathcal{P} = \{p_\alpha\}_\alpha$ be any family of polynomials in R_n. The zero set of $\mathcal{P}$ is

$$Z(\mathcal{P}) = \{(z_1, ..., z_n) \in k^n : p_\alpha(z_1, ..., z_n) = 0, \forall \alpha\}$$

Then $Z(\mathcal{P}) = \bigcap_\alpha Z(p_\alpha)$. The closed sets in k^n are by definition the collection of $Z(\mathcal{P})$ as $\mathcal{P}$ varies over all families of polynomials in R_n

Lemma. $(k^n, \{Z(\mathcal{P})\})$ is a topological space

Proof. $\emptyset = Z(1)$ for the constant polynomial 1, and $k^n = Z(0)$. Moreover $\cap_i Z(\mathcal{P}_i) = Z(\cup_i \mathcal{P}_i)$ □

Example. For $n = 1$ we have that $\mathcal{P} = \{p_\alpha\}_\alpha$ where $p_\alpha \in k[x]$, then if at least one of these polynomials is nonconstant $Z(p_\alpha)$ is a finite set, since algebraically closed fields are st each nonconstant polynomial has the number of roots equal to the degree.

Note that the Zariski topology is not Hausdorff: In the Zariski topology we know that the nontrivial open sets are complements of finite sets, and thus the intersection of two open sets is nontrivial, since the intersection must be open.

Definition (Affine Variety)**.** A closed subset of k^n in the Zariski topology

For any family $\mathcal{P} \subset R_n$ let $I(\mathcal{P})$ be the ideal generated by elements from $\mathcal{P}$. Then $Z(\mathcal{P}) = Z(I(\mathcal{P}))$ since sums of polynomials in $Z(\mathcal{P})$ will be zero, and constant multiplies are also sharing the same zero sets.

More importantly: $Z(I(\mathcal{P})) = Z(\sqrt{I(\mathcal{P})})$ for the radical ideal. Recall that the radical of an ideal $I \subset S$ is the ideal $\sqrt{I} = \{x \in S : x^n \in S, \text{for some } n > 0\}$, and $I \subset \sqrt{I}$ always holds clearly.

Theorem (Hilbert Basis Theorem)**.** If R is Noetherian then so is $R[x_1, ...x_n]$ (Noetherian means every ideal is finitely generated.)

Definition. Let $V \subset k^n$ be an affine variety. A set function $f : V \to k$ is called regular if there is a polynomial $p_f \in R_n$ such that $f(\bar{z}) = p_f(\bar{z})$ for $\bar{z} \in k^n$.

If p_f 'represents' f in the above sense then for any $g \in I(V)$, $p_f + g$ also represents f. Observe that $I(V)$ is a radical ideal.

Now we explore the relation between all these notions. We started with $\mathcal{P}$ and are able to get $Z(\mathcal{P}) = Z(I(\mathcal{P})) = Z(\sqrt{I(\mathcal{P})})$. We can also start with $\mathcal{P}$ and go to $I(\mathcal{P})$, and then to $\sqrt{I(\mathcal{P})}$. Now we take $V(\sqrt{I(\mathcal{P})}) = V(\mathcal{P})$ for a variety V. We ask a natural question:

What is the relation between $\sqrt{I(\mathcal{P})}$ and $I(V(\sqrt{I(\mathcal{P})}))$
It will turn out that these are equal.

Notice here we see a connection between geometry and algebra: On the one hand we have the set of varieties in k^n, a geometric object, and the set of radical ideals in R_n, an algebraic object. In fact this is a bijection:

$$\text{Set of varieties in } k^n \leftrightarrow \text{Set of radical ideals in } R_n$$
$$V \to I(V)$$
$$Z(I) \leftarrow I$$

Note that this only works for radical ideals, if one wanted to talk about general ideals we need to replace varieties with something more general, Grothendieck's answer was Schemes.

We say a bit more. The ideal $I(X)$ for any set $X \subset k^n$ is always a radical ideal, and so the operation $I(-)$ sends an affine variety to a radical ideal. The map $V(-)$ sends a radical ideal to an affine variety.

Proposition (Nullstellensatz)**.** a) For any affine variety $X \subset k^n$, $V(I(X)) = X$
b) For any ideal $J \subset k[x_1, ..., x_n]$, $I(V(J)) = \sqrt{J}$

Definition. Let $k[V]$ be the collection of all regular functions on $V \to k$, then it's clearly a ring under pointwise addition and multiplication.

Theorem. $k[V] \cong R_n/I(V)$ as k-algebras

Proof. First isomorphism theorem ☐

Let X, Y be two affine varieties $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$.

Definition. A set function $f : X \to Y$ is called regular or algebraic if $\bar{f} : X \to Y$ composed with projections $p_i : \mathbb{A}^m \to k$ $(z_1, .., z_m) \mapsto z_i$ is regular for all i. A regular function $f : X \to Y$ is an isomorphism if there exists an inverse map $f^{-1} : Y \to X$

Lemma. Any composition of regular functions is algebraic

Corollary. Let $f : X \to Y$ be a regular map between affine varieties, then we get a k-algebra homomorphism $f^* : k[Y] \to k[X]$ where $k[Y]$ is the k-algebra of regular functions on Y

Proof. $(f*)(\varphi) = \varphi \circ f : X \to k$, multiplication and addition are pointwise. ☐

Why this? A consequence of Nullstellesatz is that polynomials and regular functions agree on $\mathbb{A}^n$. If $f, g \in k[x_1, ..., x_n]$ are two polynomials defining the same function, i.e. $f(x) = g(x)$ for all $x \in \mathbb{A}^n$, then $f - g \in I(\mathbb{A}^n)$ the ideal of $\mathbb{A}^n$, but $I(\mathbb{A}^n) = 0$ so $f = g$ in $k[x_1, ..., x_n]$, so another way of saying this is that two polynomials f, g define the same polynomial on the affine variety X (regular) function if and only if $f - g \in I(X)$. So we get an equivalent but (imo) more clear definition:

Definition. Let $X \subset \mathbb{A}^n$ be an affine variety. A regular function is a map $X \to k$, $x \mapsto f(x)$ for some $f \in k[x_1, ..., x_n]$. The ring of all regular functions is called the coordinate ring and is $k[X] = k[x_1, ..., x_n]/I(X)$

============MORE EXPOSITION AND EXAMPLES======================MAXIMAL IDEALS CORRESPOND TO POINTS===========

Note: Coordinate rings are k-algebras (a k-vector space where the ring multiplication if k-bilinear)

Here's a question: Is any k-algebra A isomorphic with $k[X]$ for some variety X? The answer is no, A has to be finitely generated, and have no nonzero nilpotent elements.

Proposition. Any finitely generated k-algebra A is isomorphic with $k[X]$ for an affine variety

Proof. Since A is finitely generated we can write $k[x_1, ..., x_n] \twoheadrightarrow A$ where $(x_1, ..., x_n) \mapsto (a_1, ..., a_n)$. Let $I = \ker \varphi \implies R_n/I \simeq A$ since A has no nonzero nilpotence I is a radical ideal, so $X = Z(I) \subset k^n$. □

Theorem. Given two affine varieties X, Y and a k-algebra homomorphism $\varphi : k[Y] \to k[X]$ there exists a regular map $f : X \to Y$ such that $f^* = \varphi$ for $f^* : k[Y] \to k[X]$

Theorem. The category of affine varieties (and regular maps between them) is isomorphic to the category of finitely generated k-algebras with no nilpotence.

Definition. A topological space is called irreducible if X cannot be written as a union of two proper closed subsets.

Example. k is irreducible under the Zariski topology.

Lemma. Any affine variety is a finite union of irreducible affine subvarieties.

Proof. By contradiction —–FILL IN—— □

Definition. The decomposition into irreducibles is called irredundant if no X_i is contained in X_j for $i \neq j$

Lemma. An irredundant decomposition of X is unique

Definition. Under an irredundant decomposition, $X = X_1 \cup \cdots \cup X_n$ we call each X_i an irreducible component.

Example. Take the affine variety x-axis $\cup\, y$-axis, then $k^2 = Z(I)$ where $I = \langle xy \rangle$. The irreducible components are the x and y axes, but they are not connected component, the variety is connected. An irreducible variety is connected.

Theorem. An affine variety is irreducible iff $k[X]$ is an integral domain

2 Rational Functions

Let X be an irreducible affine variety, so $k[X]$ is an integral domain, and let $k(X)$ be the fraction field. An element of $k(X)$ is of the form $\frac{P(x)}{Q(x)}$ where $p, q \in k[X]$, called a rational function.

Definition. A function defined on an open subset of X is called a rational function if for any $x \in X$ we can find an element $f_x \in k(X)$ such that $f = f_x$ in a neighborhood of x

Example. Any function $f \in k(X)$ is a rational function on X defined on these points where Q does not vanish, so $f_x = \frac{P(x)}{Q(x)}$

——————————-INCLUDE SOME MORE EXAMPLES AND EXPOSITION OF THESE THINGS————

Theorem. Let f be a rational function on an irreducible affine variety X defined everywhere, then $f \in k[X]$. Said another way: If f is a rational function on an affine variety defined everyone, then f is regular.

So really rational functions are local objects, and if the happen to be global then they're regular functions.

Proof. Locally for any $x \in X$ $f = P_x/Q_x$ where $Q_x(x) \neq 0$. So $I = \langle Q_x \rangle_{x \in X}$ is some ideal. as this is a Noetherian ring we know $I = \langle Q_{x_i} \rangle_{i=1}^n$ (finitely generated), thus $Z(I) = \emptyset$ since $Q_x(x) \neq 0$. By Hilbert's basis theorem we get $\sqrt{I} = (1)$ and so $\sqrt{I} = R$, thus $1^n \in I$ so $1 \in I$, hence $I = R$. Therefore $1 = \sum R_{x_i} Q_{x_i}$. We claim that $f = \sum R_{x_i} P_{x_i}$. Now $R_{x_i}, P_{x_i}, Q_{x_i} \in k[X]$ so $\frac{P_{x_i}}{Q_{x_i}} = \sum R_{x_i} P_{x_i} \iff P_{x_j} = \sum R_{x_i} P_{x_i} Q_{x_j}$ in a neighborhood of x_j. So

$$\frac{P_{x_i}}{Q_{x_i}} = \frac{P_{x_j}}{Q_{x_j}} \iff P_{x_i} Q_{x_j} = P_{x_j} Q_{x_i} = \left(\sum R_{x_i} Q_{x_i} \right) P_{x_j} = 1 \cdot P_{x_j}$$

□

For this to work we need local checking at every point. We want to go from global objects to local objects. Affine varieties are global objects, and we want to extend this to local objects. Consider the following object.

$\mathbb{P}_k^n = k^{n+1} \setminus \{0\}/k^*$, where $v \sim w \iff v = zw$ for some $z \in k^*$, this is the same as the space of lines through 0 in k^n. Why do we care about this? Compactness. If $k = \mathbb{C}$ then $\mathbb{P}_{\mathbb{C}}^n$ has the Hausdorff topology from the quotient or analytic topology, this is compact, unlike $\mathbb{C}^n$.

Now consider what the Zariski topology on $\mathbb{P}^n$ is. If we have $P \in k[t_0, ..., t_n]$ let $\vec{z} \in Z(P)$, then $\vec{z} \in k^{n+1}$. If $\lambda \vec{z} \in Z(P)$ for all λ then $p = p_0 + \cdots + p_\alpha$ where p_i is a homogenous polynomial of total degree i. Then

$$\begin{aligned} p(\lambda \vec{z}) &= p_0(\lambda \vec{z}) + \cdots + p_\alpha(\lambda \vec{z}) \\ &= \lambda^0 p_0(\vec{z}) + \cdots \lambda^d p_\alpha(\vec{z}) = 0 \end{aligned}$$

If and only if $p_i(\vec{z}) = 0$ for all i. So $Z(P) = Z(P_0, .., P_d)$ so really just need to consider homogenous polynomials.

An ideal $I \subset k[x_0, ..., x_n]$ is called homogenous if whenever $P \in I$ all its components belong to I iff I is generated by certain homogenous polynomials.

Definition. The Zariski topology on $\mathbb{P}^n$ is gotten by taking $X \in \mathbb{P}^n$ and saying this is closed iff there is a homogenous ideal $I \subset k[x_0, .., x_n]$ such that $X = Z(I)$

Theorem. This gives a topology on projective space

Definition. A set C is called locally closed in a topological space X if $C = U \cap K$ where U is open in X and K is closed in X OR C is locally closed if its closed in some open subset.

Now we have affine varieties, but using the above we get a more general class of objects: Varieties

Definition. A variety is any locally closed set X in $\mathbb{P}^n$ under the Zariski topology.

If X is closed in $\mathbb{P}^n$ then it's called a projective variety. It should be noted that often the word "variety" means quasi-projective.

Proposition. Any affine variety is a variety

Proof. ——————FILL IN—————— □

Corollary. Any affine variety is a variety

Definition. Regular functions on a variety X. Let X be a variety (locally closed in $\mathbb{P}^n$), then if $f : X \to k$ is a map, f is called regular if for any $x \in X$ we can find a rational function P_x/Q_x both homogenous polynomials of the same degree such that $f = P_x/Q_x$ and $Q_x(x) \neq 0$

Definition. Let X, Y be varieties, $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$, and $\mathbb{P}^m$ has the open cover U_i, then $f : X \to Y$ is regular if $f^{-1}(U_i)$ is open in X and $f^{-1}(U_i) \to U_i$ is regular.

Theorem. Let X be an affine variety, then X is irreducible if and only if $k[X]$ is an integral domain

Proof. ————-FILL IN————- □

3 Vector Bundles and Divisors

4 Sheaf Theory

5 Cohomology of Sheaves

6 Serre Duality and Riemann-Roch

7 Thoughts

7.1 Functions

Regular and rational functions gave me headaches since there seem to be lots of definitions. All in all though they convey the same idea: A rational function $X \to k$ is just a function given by some kind of algebraic equation, usually a polynomial. These are global objects, defined for every point of X. Rational maps, as the name suggests are ratios of polynomials, but these are more local objects, being defined not everywhere but usually for a "large enough" space, like a dense open subset.

Regular and rational MAPS are maps between varieties given by tuples of regular and rational functions, respectively.

8 Homeworks and Solutions

9 HW 1

1. Pages 32-34: 1,6, 9, 10, 12, 13, 15, 16
   2. Pages 40-41: 1, 4, 6, 7.

10 32-34

10.1 1

10.2 6

10.3 9

10.4 10

10.5 12

10.6 13

10.7 15

10.8 16

11 40-41

11.1 1

11.2 4

11.3 6

11.4 7

12 HW 2

Pages 53-54: 1, 2, 3, 4, 5, 11.
Pages 66-67: 1, 4, 5, 9,11

13 53-54

13.1 1

An affine variety U is irreducible if and only if its projective closure $\bar{U}$ is irreducible

U reducible means that $U = (A \cap U) \cup (B \cap U)$ for A, B closed proper subsets (closed wrt U). As A, B closed in U we know that A, B are closed in $\bar{U}$, and these are proper subsets in U hence in $\bar{U}$. So $\bar{U} = (A \cap \bar{U}) \cup (B \cap \bar{U})$, since neither A, B contain $\bar{U}$.

Question: Closed wrt U??? Thus need to show these are closed proper with wrt $\bar{U}$

13.2 2

Associate with any affine variety U $\subset \mathbb{A}^n$ 0 its projective closure U in $\mathbb{P}^n$. Prove that this defines a one-to-one correspondence between the affine subvarieties of $\mathbb{A}^n$ 0 and the projective subvarieties of $\mathbb{P}^n$ with no components contained in the hyperplane S0 = 0.

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13.3 3

Show $X = \mathbb{A}^2 \setminus (0,0)$ is not isomorphic to any affine variety

This is a bit tricky. Here $k[\mathbb{A}^n] \subset k[X]$, since 1=$f_1x + f_2y$ where $f_1, f_2 \in k[X]$, and $\mathbb{A}^2 \setminus (\mathbb{A}^1 \times \{0\}) = A^1 \times k^*$ and this is affine. The main idea is that there is an injective function $k[\mathbb{A}^2] \hookrightarrow k[\mathbb{A}^2 \setminus (0,0)]$ want this to be iso. The key observation is that a poly $P \in k[t_1, t_2]$ then $Z(P)$ cannot be a single point....

13.4 4

Prove that any quasiprojective variety is open in its projective closure

Let $X \subset \mathbb{P}^n$ be a quasi-projective variety, so X is an locally closed subset of $\mathbb{P}^n$ with the Zariski topology. We know that $X = Y \cap Z$ where Y is open and Z is closed inside $\mathbb{P}^n$. Moreover we get the projective closure $\bar{X}$, where $X \subset \bar{X}$.

13.5 5

Show any rational $\varphi : \mathbb{P}^1 \to \mathbb{P}^n$ is regular

If φ is a rational map then by definition it's given by n+1 forms $(x : y) \mapsto (\varphi_0(x,y) : \cdots : \varphi_n(x,y))$ where all φ_i's are homogenous of the same degree. At least one of these must not vanish, and if they all vanish we can remove the common factor. If we can define φ such that $(f_0 : \cdot : f_n)$ st all are regular at $x \in X$ and not all zero then f is regular at x. But we know that φ is such a map, not all elements are regular at (x,y) since we can remove the common factor. Moreover we can define a rational map to be a regular map at some open $U \subset \mathbb{P}^1$, thus we get the above, since each φ_i is regular, it's a regular map, and it's not all zero in the open set U, hence we can extend to a regular map on all $\mathbb{P}^1$.

13.6 11

Prove that the variety $\mathbb{P}^n \setminus X$, where X is an plane conic is affine

X is the zero set of deg 2 homogenous poly. $X = Z(P_2)$. Let L be a linear form, so degree 1 hom. poly. $\mathbb{P}^n \setminus Z(L)$ is affine. since if $L = x_0$ where $(x_0, ..., x_n)$ coordinates, then $\mathbb{P}^n \setminus Z(x_0) \simeq \{1, x_1, ..., x_n\} = \mathbb{A}^n$. Take t_1, t_2, t_2 to get a basis of $S^2(k^3)$, in terms of these coordinates we linearize this. Similary for P_2 and $\mathbb{P}^2 \setminus X \hookrightarrow \mathbb{P}(S^2(k^3)) \setminus L$, the map sends $t_1, t_2, t_2 \mapsto t_1^2, t_2^2, t_3^2$ this is the Segre embedding.....

14 66-67

14.1 1

Prove that the Segre variety $\varphi(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^N$,where $N = (n+1)((m+1)-1)$ is not contained in any linear subspace strictly smaller than the whole of $\mathbb{P}^N$

14.2 4

Let $X = \mathbb{A}^2 \setminus x$ prove that X is not isomorphic to an affine or projective variety

Similar to question above. Any regular function on an irreducible projective variety is constant, there are lots of nonconstant reg functions on X.

14.3 5

Let $X = \mathbb{P}^2 \setminus x$ prove that X is not isomorphic to an affine or projective variety

X is not compact so not projective, $\mathbb{P}^2 \setminus x \hookrightarrow \mathbb{P}^2$, we know the image of a projective variety under any map is closed, but this can't be closed as $\mathbb{P}^2 \setminus x$ is open, as we remove a closed pt, then we'd have $\mathbb{P}^2 = x \cup \mathbb{P}^2 \setminus x$, so not irreducible. Not affine??

14.4 9

Prove that any intersection of affine open subsets is affine

Use that for closed X, Y the intersection $X \cap Y \simeq X \times Y \cap \Delta$ is closed. Take open sets in X, Y use this fact to get the answer.

14.5 11

Let $f : X \to Y$ be a regular map of affine varieties. Prove that the inverse image of a principal affine open set is a principal affine open set

Principal affine open means $X \setminus Z(f)$ for a function. $h \circ f$, then the zero sets corresponds.

15 HW 3

Pages 80-81: 1, 2, 5, 12
Pages 95-97: 1, 4, 6, 7, 8, 10, 12, 18.

16 80-81

16.1 1

Let $L \subset \mathbb{P}^n$ be an $(n-1)$ dimensional linear subspace $X \subset L$ an irreducible closed variety and y a point in $\mathbb{P}^n \setminus L$. Join to y all points $x \in X$ by lines and denote by Y the set of

points lying on all these lines, that is, the cone over X with vertex y. Show Y is an irreducible projective variety, and dim Y = dim X+1

???????????? What????????????????????

16.2 2

Let $X \subset \mathbb{A}^3$ be the reducible curve whose components are the 3 coordinate axes, prove that the ideal I_X cannot be generated by 2 elements

16.3 5

Prove any finite set of points $S \subset \mathbb{P}^2$ can be defined by two equations

Induction?

16.4 12

——— ???????????????????

17 95-97

17.1 1

Prove that the local ring $\mathcal{O}_{X,x}$ of a point x of an irreducible variety X is the union of alll $k(X)$ of all the rings $k[U]$ for a neighborhood U of x.

17.2 4

17.3 6

Determine the local ring of $(0,0)$ of the curve $xy(x-y)=0$. Prove that this curve is not isomorphic to the curve of 3 coordinate axes

17.4 7

Prove if $x \in X, y \in Y$ are nonsingular points then $(x,y) \in X \times Y$ is nonsingular

17.5 8

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17.6 10

Prove that if a hypersurface X has two singular points then the line joining them is contained in X

17.7 12

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17.8 18

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18 HW 5

19 166-167

2,4,12,15,17

20 Hartshorne Exercises

21 Chpt II.1

1.2,1.3,1.6,1.7,1.8,1.10,1.11,1.12,1.14,1.15,1.17

22 Chpt II.2

2.1,2.2,2.3,2.10,2.13,2.14,2.17

23 Chpt II.3

24 Chpt II.5

5.1 5.3,5.4,5.6,5.16,5.18

25 Chpt II.6

26 Chpt III.2

2.1,2/2.2.3,2.4,2.6,2.7

27 Chpt III.3

28 Chpt III.4